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ZERO-FREE REGIONS FOR POLYNOMIALS WITH SPECIAL COMPLEX COEFFICIENTS

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ABSTRACT

In this paper we can extend the well-known result Eneström-Kakeya theorem by relaxing the hypothesis in several ways and obtain zero-free regions for polynomials with special complex coefficients and there by present some interesting generalizations and extensions of the Eneström-Kakeya Theorem.

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Keywords: *Zeros of polynomial, Polar Derivatives, Eneström-Kakeya theorem.*

1. INTRODUCTION

The well-known Results Eneström-Kakeya theorem [2, 4] in theory of the distribution of zeros of polynomials is the following.

Theorem 1.1: Let P(z) = $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n* such that $0 < a_0 \leq a_1 \leq a_2 \leq, \dots, \leq a_n$ then all the zeros of P(z) lie in $|z| \leq 1$.

Applying the above result to the polynomial $z^n P(\frac{1}{z})$ we get the following result:

Theorem 1.2: If P(z) = $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_n \leq a_{n-1} \leq a_{n-2} \leq, \dots, \leq a_0$ then P(z) does not vanish in $|z| < 1$.

In the literature [1, 3, 5-9], there exist several extensions and generalizations of the Eneström-Kakeya Theorem. Recently B. A. Zargar [11] proved the following results:

Theorem 1.3: If P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that for some $k \geq 1$, $0 < a_n \leq a_{n-1} \leq a_{n-2} \leq, \dots, \leq a_0$ then P(z) does not vanish in the disk $|z| < \frac{1}{2k-1}$.

Theorem 1.4: If P(z)= $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree such that for some real number $\rho \geq 0$

$0 < a_0 \leq a_1 \leq a_2 \leq, \dots, \leq a_{n-1} \leq a_n + \rho$, then P(z) does not vanish in the disk $|z| < \frac{1}{2(a_n+\rho)-a_o}$.

The following results due to P. Ramulu [10].

Theorem 1.5: Let P(z) = $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$ $\rho \geq 0$, $a_m \neq 0$, $a_n - \rho \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq k a_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0$ then all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{2k(a_m+|a_m|)-(a_0+2|a_m|+a_n)+a_n+2\rho}$.

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Theorem 1.6: Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $0 < r \leq 1$, $\rho \geq 0$, $a_m \neq 0$, $a_n + \rho \geq a_{n-1} \geq, \dots, \geq a_{m+1} \geq ra_m \leq a_{m-1} \leq, \dots, \leq a_1 \leq a_0$ then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{a_0+2|a_m|-2r(a_m+|a_m|)+a_n+|a_n|+2\rho}.$$

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results.

2. MAIN RESULTS

Theorem 2.1: Let P(z) = $\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some k ≥ 1 , $\xi \geq 0$, $a_m \neq 0$, $a_n - \xi \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0$ and for some $t \geq 1$, $\eta \geq 0$, $b_m \neq 0, b_n - \eta \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq tb_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Corollary 2.2: Let P(z) = $\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for somek ≥ 1 , $\xi \geq 0$, $a_m \neq 0$, $0 < a_n - \xi \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0 > 0$ and $t \geq 1, \eta \geq 0$, $b_m \neq 0, 0 < b_n - \eta \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq tb_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0 > 0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[(2k-1)a_m + (2t-1)b_m + \xi + \eta] - (a_0 + b_0)}.$$

Corollary 2.3: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some k $\geq 1, \xi \geq 0$, $a_m \neq 0$, $a_n - \xi \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0$ and $b_n \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq b_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[k(|a_m| + a_m) + b_m - |a_m| + \xi] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Corollary 2.4: Let P(z) = $\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that $a_n \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq a_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0$ and for some $t \geq 1, \eta \geq 0$, $b_m \neq 0, b_n - \eta \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq tb_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[a_m + t(|b_m| + b_m) - |b_m| + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Corollary 2.5: Let P(z) = $\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for somek ≥ 1 , $\xi \geq 0$, $a_m \neq 0$, $a_n - \xi \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0$ and $b_m \neq 0, b_n - \xi \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq kb_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[k(|a_m| + |b_m| + a_m + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Corollary 2.6: Let P(z) = $\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $a_n \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq a_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0$ and $b_n \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq b_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[a_m + b_m] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Corollary 2.7: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $0 < a_n \leq a_{n-1} \leq, \dots, \leq a_{m+1} \leq a_m \geq a_{m-1} \geq, \dots, \geq a_1 \geq a_0 > 0$ and $0 < b_n \leq b_{n-1} \leq, \dots, \leq b_{m+1} \leq b_m \geq b_{m-1} \geq, \dots, \geq b_1 \geq b_0 > 0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[a_m + b_m] - (a_0 + b_0)}.$$

Remark 2.8: By taking $a_i > 0$ *and* $b_i > 0$ *for* $i = 0,1,2, \dots, n$, in Theorem 2.1, it reduces to Corollary 2.2.

Remark 2.9: By taking $\eta = 0$ *and* $t = 1$ in Theorem 2.1, it reduces to Corollary 2.3.

Remark 2.10: By taking $\xi = 0$ *and* $k = 1$ in Theorem 2.1, it reduces to Corollary 2.4

Remark 2.11: By taking $\eta = \xi \; and \; t = k$ in Theorem 2.1, it reduces to Corollary 2.5.

Remark 2.12: By taking $\eta = \xi = 0 \; and \; k = t = 1$ in Theorem 2.1, it reduces to Corollary 2.6.

Remark 2.13: By taking $\eta = \xi = 0\,, k = t = 1 \; and \; a_i > 0, b_i > 0 \; for i = 0,1,2, \dots, n$, in Theorem 2.1, it reduces to Corollary 2.7.

Remark 2.14: By taking $b_i = 0 \; and \; \xi = \rho$ in Theorem 1, it reduces to Theorem 1.5.

Theorem 2.15: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $0 < \tau \le 1,\; k \ge 1,\; \xi \ge 0,\; a_m \ne 0,\; a_n + \xi \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge \tau a_m \le a_{m-1} \le, \dots, \le a_1 \le k a_0$ and for some $0 < \mu \le 1,\; t \ge 1, \eta \ge 0,\; b_m \ne 0, b_n + \eta \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge \mu b_m \le b_{m-1} \le, \dots, \le b_1 \le t b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{k(|a_0| + a_0) + t(|b_0| + b_0) + X + |a_n| + |b_n| + a_n + b_n - (|a_0| + |b_0|)},$$

where $X = 2[|a_m| + |b_m| + \xi + \eta - \tau(|a_m| + a_m) - \mu(|b_m| + b_m)]$.

Corollary 2.16: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $0 < \tau \le 1,\; k \ge 1,\; \xi \ge 0,\; a_m \ne 0,\; 0 < a_n + \xi \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge \tau a_m \le a_{m-1} \le, \dots, \le a_1 \le k a_0 > 0$ and for some $0 < \mu \le 1,\; t \ge 1, \eta \ge 0,\; b_m \ne 0, 0 < b_n + \eta \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge \mu b_m \le b_{m-1} \le, \dots, \le b_1 \le t b_0 > 0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[k a_0 + t b_0 - \tau a_m - \mu b_m + a_n + b_n + \xi + \eta] + [a_m + b_m - a_0 - b_0]}.$$

Corollary 2.17: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $0 < \tau \le 1,\; k \ge 1,\; \xi \ge 0,\; a_m \ne 0,\; a_n + \xi \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge \tau a_m \le a_{m-1} \le, \dots, \le a_1 \le k a_0$ and for some $b_n \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge b_m \le b_{m-1} \le, \dots, \le b_1 \le b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{k(|a_0| + a_0) + b_0 + 2[|a_m| + \xi - \tau(|a_m| + a_m) - b_m] + |a_n| + |b_n| + a_n + b_n - |a_0|}.$$

Corollary 2.18: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $a_n \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge a_m \le a_{m-1} \le, \dots, \le a_1 \le a_0$ and for some $0 < \mu \le 1,\; t \ge 1, \eta \ge 0,\; b_m \ne 0, b_n + \eta \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge \mu b_m \le b_{m-1} \le, \dots, \le b_1 \le t b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{t(|b_0| + b_0) + 2[|b_m| + \eta - a_m - \mu(|b_m| + b_m)] + |a_n| + |b_n| + a_n + b_n - |b_0|}.$$

Corollary 2.19: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $0 < \tau \le 1,\; k \ge 1,\; \xi \ge 0,\; a_m \ne 0,\; a_n + \xi \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge \tau a_m \le a_{m-1} \le, \dots, \le a_1 \le k a_0$ and for some $b_m \ne 0, b_n + \xi \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge \tau b_m \le b_{m-1} \le, \dots, \le b_1 \le k b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{k(|a_0| + |b_0| + a_0 + b_0) + |a_n| + |b_n| + a_n + b_n - (|a_0| + |b_0|)},$$

where $X_1 = 2[|a_m| + |b_m| + 2\xi - \tau(|a_m| + a_m + |b_m| + b_m)]$.

Corollary 2.20: Let P(z) $= \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some , $a_n \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge a_m \le a_{m-1} \le, \dots, \le a_1 \le a_0$ and for some $b_n \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge b_m \le b_{m-1} \le, \dots, \le b_1 \le b_0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{(|a_0| + |b_0| + a_0 + b_0) - (a_m + b_m) + |a_n| + |b_n| + a_n + b_n - (|a_0| + |b_0|)}.$$

Corollary 2.21: Let P(z) $= \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that $0 < a_n \ge a_{n-1} \ge, \dots, \ge a_{m+1} \ge a_m \le a_{m-1} \le, \dots, \le a_1 \le a_0 > 0$ and $0 < b_n \ge b_{n-1} \ge, \dots, \ge b_{m+1} \ge b_m \le b_{m-1} \le, \dots, \le b_1 \le b_0 > 0$, then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{(a_0 + b_0) - (a_m + b_m) + |a_n| + |b_n| + 2(a_n + b_n}.$$

Remark 2.22: By taking $a_i > 0$ and $b_i > 0$ $for i = 0,1,2,\ldots,n,$ in Theorem 2, it reduces to Corollary 2.16.

Remark 2.23: By taking $\eta = 0$ and $t = \mu = 1$ in Theorem 2.15, it reduces to Corollary 2.17.

Remark 2.24: By taking $\xi = 0$ and $k = \tau = 1$ in Theorem 2.15, it reduces to Corollary 2.18.

Remark 2.25: By taking $\eta = \xi$ $, \mu = \tau$ and $t = k$ in Theorem 2.15, it reduces to Corollary 2.19.

Remark 2.26: By taking $\eta = \xi = 0$ and $\mu = \tau = k = t = 1$ in Theorem 2.15, it reduces to Corollary 2.20.

Remark 2.27: By taking $\eta = \xi = 0$ and $\mu = \tau = k = t = 1$ and $a_i > 0, b_i > 0$ $for i = 0,1,2,\ldots,n,$ in Theorem 2.15, itreduces to Corollary 2.21.

Remark 2.28: By taking $b_i = 0$, $\tau = r$ and $\xi = \rho$ in Theorem 2.15, it reduces toTheorem 1.6.

Theorem 2.29: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for somek ≥ 1 $, \xi \geq 0,$ $a_m \neq 0,$ $a_n - \xi \leq a_{n-1} \leq, \ldots, \leq a_{m+1} \leq k a_m \geq a_{m-1} \geq, \ldots, \geq a_1 \geq a_0$ and for some $0 < \mu \leq 1,$ $t \geq 1, \eta \geq 0,$ $b_m \neq 0, b_n + \eta \geq b_{n-1} \geq, \ldots, \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq, \ldots, \leq b_1 \leq t b_0,$ then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{t(|b_0| + b_0) + (a_0 + |b_0|) + X_2 + |a_n| + |b_n| - a_n + b_n},$$

where $X_2 = 2[k(|a_m| + a_m) - |a_m| + |b_m| - \mu(|b_m| + b_m) + \xi + \eta].$

Theorem 2.30: Let P(z) $=\sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ such that for some $0 < \tau \leq 1,$ $k \geq 1,$ $\xi \geq 0,$ $a_m \neq 0,$ $a_n + \xi \geq a_{n-1} \geq, \ldots, \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq, \ldots, \leq a_1 \leq k a_0$ and for some$t \geq 1, \eta \geq 0,$ $b_m \neq 0, b_n - \eta \leq b_{n-1} \leq, \ldots, \leq b_{m+1} \leq t b_m \geq b_{m-1} \geq, \ldots, \geq b_1 \geq b_0,$ then all the zeros of P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{k(|a_0| + a_0) - (|a_0| + b_0) + X_3 + a_n - b_n + |a_n| + |b_n|},$$

where $X_3 = 2[|a_m| - |b_m| - \mu(|a_m| + a_m) + t(|b_m| + b_m) + \xi + \eta].$

3. Proofs of the Theorems

Proof of the Theorem 2.1:

Let P(z)= $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_{m+1} z^{m+1} + \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0$

Let Consider the polynomial J(z) = z^nP($\frac{1}{z}$)

And R(z) = (z-1)J(z) so that

Then R(z) = (z-1)$(\alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{m-1} z^{m-1} + \alpha_m z^m + \alpha_{m+1} z^{m+1} + \cdots + \alpha_{n-1} z + a_n)$

$$= \alpha_0 z^{n+1} - \{(\alpha_0 - \alpha_1) z^n + (\alpha_1 - \alpha_2) z^{n-1} + \cdots + (\alpha_{m-1} - \alpha_m) z^{n-m+1} + (\alpha_m - \alpha_{m+1}) z^{n-m} + \cdots + \alpha - \alpha_n) z + \alpha_n\}$$

$$= \alpha_0 z^{n+1} - \{(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n\} - i\{(b_0 - b_1) z^n + (b_1 - b_2) z^{n-1} + \cdots + (b_{m-1} - b_m) z^{n-m+1} + (b_m - b_{m+1}) z^{n-m} + \cdots + (b_{n-1} - b_n) z + b_n\}$$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|\mathbf{z}|^{\mathbf{n-i}}} < for\ i = 0,1,2,\ldots,n-1.$ Now

$$|R(z)| \geq |\alpha_0||z|^{n+1} - [\{|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^n + |a_m - a_{m+1}||z|^n + \cdots + |a_{n-1} - a_n||z| + |a_n|\} + \{|b_0 - b_1||z|^n + |b_1 - b_2||z|^{n-1} + \cdots + |b_{m-1} - b_m||z|^n + |b - b_{m+1}||z|^n + \cdots + |b_{n-1} - b_n||z| + |b_n|\}]$$

$$|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}$$

$$\geq |\alpha_0||z|^n[|z| - \frac{1}{|\alpha_0|}\{(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_1 - a_2|}{|z|^2} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}) + (|b_0 - b_1| + \frac{|b_0 - b_1|}{|z|} + \frac{|b_1 - b_2|}{|z|^2} + \cdots + \frac{|b_{m-1} - b_m|}{|z|^{m-1}} + \frac{|b_m - b_{m+1}|}{|z|^m} + \cdots + \frac{|b_{n-2} - b_{n-1}|}{|z|^{n-2}} + \frac{|b_{n-1} - b_n|}{|z|^{n-1}} + \frac{|b_n|}{|z|^n})\}]$$

$$\geq |\alpha_0||z|^{n+1}[|z| - \frac{1}{|\alpha_0|}\{(|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{m-1} - ka_m + ka_m - a_m| + |a_m - ka_m + ka_m - a_{m+1}| + \cdots + |a_{n-2} - a_{n-1}| + |a_{n-1} - \xi + \xi - a_n| + |a_n|) + (|b_0 - b_1| + |b_1 - b_2| + \cdots + |b_{m-1} - tb_m + tb_m - b_m| + |b_m - tb_m + tb_m - b_{m+1}| + \cdots + |b_{n-2} - b_{n-1}| + |b_{n-1} - \eta + \eta - b_n| + |b_n|)\}]$$

$$\geq |\alpha_0||z|^{n+1}[|z| - \frac{1}{|\alpha_0|}\{((a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (ka_m - a_{m-1}) + 2(k-1)|a_m| + (ka_m - a_{m+1}) + \cdots + (a_{n-2} - a_{n-1}) + (a_{n-1} + \xi - a_n) + \xi + |a_n|) + ((b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + \cdots + (tb_m - b_{m-1}) + 2(t-1)|b_m| + (tb_m - b_{m+1}) + \cdots + (b_{n-2} - b_{n-1}) + (b_{n-1} + \eta - b_n) + \eta + |b_n|)\}]$$

$$\geq |\alpha_0||z|^{n+1}[|z| - \frac{1}{|\alpha_0|}\{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)\}]$$

$$> 0$$

if $|z| > \frac{1}{|\alpha_0|}\{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)\}$.

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|\alpha_0|}\{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)\}.$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|\alpha_0|}\{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)\}.$$

Since P(z) = z^nJ($\frac{1}{z}$) it follows, by replacing z by $\frac{1}{z}$,

Then all the zeros of P(z) lie in

$$|z| \geq \frac{|\alpha_0|}{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|\alpha_0|}{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

This completes the proof of the Theorem 2.1.

Proof of the Theorem 2.15: Proof of the Theorem 2.15 is similar to that the proof of Theorem 2.1.

Proof of the Theorem 2.29: Proof of the Theorem 2.29 is similar to that the proof of Theorem 2.1.

Proof of the Theorem 2.30: Proof of the Theorem 2.30.is similar to that the proof of Theorem 2.1.

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